Modular frames for Hilbert C*-modules and symmetric approximation of frames

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ABSTRACT

We give a comprehensive introduction to a general modular frame construction in Hilbert C*-modules and to related linear operators on them. The Hilbert space situation appears as a special case. The reported investigations rely on the idea of geometric dilation to standard Hilbert C*-modules over unital C*-algebras that admit an orthonormal modular Riesz basis. Interrelations and applications to classical frame theory are indicated. Resorting to frames in Hilbert spaces we discuss some measures for pairs of frames to be close to one another. In particular, the existence and uniqueness of the closest (normalized) tight frame to a given frame is investigated. For Riesz bases with certain restrictions the set of closest tight frames often contains a multiple of its symmetric orthogonalization.

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1. INTRODUCTION

The inner structure of Hilbert spaces is easily described by fixing a basis, orthonormalizing it and working with the coordinates of every element with respect to the latter. Considering finite-dimensional Hilbert spaces as free $\mathbb{C}$-modules one could ask whether similar generating sets of finitely generated projective C*-modules can be indicated which characterize the modules up to isomorphism. Unfortunately, there are two obstacles: projective C*-modules need not to be free in general, and there does not exist any general notion of 'C*-linear independence' of sets of generators because of the existence of zero-divisors in any non-trivial C*-algebra. Beside these circumstances one does not know any canonical method to replace the process of Gram-Schmidt or symmetric orthogonalization of bases in the situation of sets of modular generators.

In 1997, Working with frames for Hilbert spaces that arise canonically in wavelet and Weyl-Heisenberg / Gabor frame theory, we got the idea to investigate modular frames of Hilbert C*-modules over unital C*-algebras as a possible replacement for the questionable analogs of bases. This class of C*-modules includes finitely generated projective ones. The resulting theory[1,2] has been encouraging because of its consistency and strength, and also because of the number of mathematical problems which can make use of it. In the field of wavelet and frame theory and its applications to signal and image processing many frames arise as the result of group actions on single functions. Extending the group to its (reduced) group C*-algebra or to its group von Neumann algebra and taking the generated frame as a generating set of a Hilbert C*-module over one of these C*-algebras we are in the context in which our concept can be applied. This point of view has been of interest e.g. to M. A. Rieffel, to O. Bratteli and P. E. T. Jørgensen, and to P. G. Casazza and M. Lammers as we know from ongoing discussions. In the literature we found other fields of applications like the description of conditional expectations of finite (Jones) index on C*-algebras ([3,4,5,6]), the analysis of Cuntz-Krieger-Pimsner algebras ([7,8]), the investigation of the stable rank of C*-algebras ([9,10]) and the search for L^2-invariants in global analysis ([11,12,13]).

The purpose of the present paper is to give a survey on our results on modular frames for Hilbert C*-modules indicating their generality and strength as well as pointing out differences to the Hilbert space frame theory and open problems. For full proofs and more details we refer to our basic publications ([1,2]). The next section we explain decomposition and reconstruction results. In the third section we deal with frame-related invariants of finitely generated projective C*-modules that characterize them up to isomorphism. the fourth section is devoted to a structure theorem on the nature of operators $\{b_i\}_i$ on a certain Hilbert space such that $\sum_i b_i^* b_i = id$. The last section is concerned with a discussion on various problems of frame approximation by (normalized) tight ones.

2. MODULAR FRAMES FOR HILBERT C*-MODULES

The concept of Hilbert C*-modules arose as a generalization of the notions 'Hilbert space', 'fibre bundle' and 'ideal'. The basic idea has been to consider modules over arbitrary C*-algebras instead of linear spaces and to allow the inner products to take values in those C*-algebras of coefficients being C*-(anti-)linear in their arguments. For the history and for comprehensive accounts we refer to the publications by E. C. Lance,[14] by N.-E. Wegge-Olsen[15] and by I. Raeburn, D. P. Williams.[16]

DEFINITION 2.1. *Let A be a (unital) C*-algebra and $\mathcal{M}$ be a (left) A-module. Suppose that the linear structures given on A and $\mathcal{M}$ are compatible, i.e. $\lambda(ax) = (\lambda a)x = a(\lambda x)$ for every $\lambda \in \mathbb{C}$, $a \in A$ and $x \in \mathcal{M}$. If there exists a mapping $\langle .,. \rangle : \mathcal{M} \times \mathcal{M} \to A$ with the properties*

  (i) *$\langle x, x \rangle \geq 0$ for every $x \in \mathcal{M}$,*

  (ii) *$\langle x, x \rangle = 0$ if and only if $x = 0$,*

  (iii) *$\langle x, y \rangle = \langle y, x \rangle^*$ for every $x, y \in \mathcal{M}$,*

  (iv) *$\langle ax, y \rangle = a\langle x, y \rangle$ for every $a \in A$, every $x, y \in \mathcal{M}$,*

  (v) *$\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ for every $x, y, z \in \mathcal{M}$,*

then the pair $\{\mathcal{M}, \langle .,. \rangle\}$ is called a (left) pre-Hilbert A-module. The map $\langle .,. \rangle$ is said to be an A-valued inner product. If the pre-Hilbert A-module $\{\mathcal{M}, \langle .,. \rangle\}$ is complete with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$ then it is called a Hilbert A-module.

Two Hilbert A-modules are unitarily isomorphic if there exists a bounded A-linear isomorphism of them which preserves the inner product values.

In case A is unital the Hilbert A-module $\mathcal{M}$ is (algebraically) finitely generated if there exists a finite set $\{x_i\}_{i \in \mathbb{N}} \subset \mathcal{M}$ such that $x = \sum_i a_i x_i$ for every $x \in \mathcal{M}$ and some coefficients $\{a_i\} \subset A$. If A is unital the Hilbert A-module $\mathcal{M}$ is countably generated if there exists a countable set $\{x_i\}_{i \in \mathbb{N}} \subset \mathcal{M}$ such that the set of all finite A-linear combinations $\{\sum_j a_j x_j\}$, $\{a_i\} \subset A$, is norm-dense in $\mathcal{M}$.

Hilbert C*-modules appear naturally in a number of situations. For example, the set of all essentially bounded measurable maps of a measure space X into a measurable field of Hilbert spaces on X becomes a Hilbert $L^\infty(X)$-module after factorization by the set of maps that are non-zero only on sets of measure zero, ([17]). A simple algebraic construction is the algebraic tensor product $A \odot H$ of a C*-algebra A and a Hilbert space H completed with respect to the norm which is derived from the A-valued inner product $\langle a \otimes h, b \otimes g \rangle = ab^* \langle h, g \rangle_H$ for $a, b \in A$, $h, g \in H$. It is denoted by $A \otimes H$. This kind of Hilbert C*-modules is very important: every finitely generated Hilbert A-module $\mathcal{M}$ can be embedded into some Hilbert A-module $A^n = A \odot \mathbb{C}^n$ for finite $n \in \mathbb{N}$ as an orthogonal summand, and every countably generated Hilbert A-module can be realized as an orthogonal summand of the Hilbert A-module $l_2(A) = A \otimes l_2$ in such a way that its orthogonal complement is isometrically isomorphic to $A \otimes l_2$ again, (Kasparov's theorem[14,15]). To name two further examples, the set of all continuous sections of a certain vector bundle over a base space X becomes naturally a finitely generated Hilbert $C(X)$-module, and conditional expectations on C*-algebras B turn these C*-algebras into pre-Hilbert A-modules over the image C*-subalgebra $A \subseteq B$. (By the way, the interrelation between vector bundles and finitely generated projective C*-modules over commutative C*-algebras is a categorical equivalence similar to Gel'fand's theorem for C*-algebras, see Serre-Swan's theorem[15].)

The reader should be aware that the theory of Hilbert C*-modules has much more exceptional examples in comparison to Hilbert space theory then one can think of. E.g. the analog of the Riesz' representation theorem for bounded A-linear functionals on Hilbert A-modules is not valid, in general. Therefore, bounded modular operators may be non-adjointable, norm-closed Hilbert A-submodules can lack the property to be direct summands, the notions of topological and orthogonal direct summands are different, and more. There are even examples of non-countably generated Banach C*-modules which can be turned into Hilbert C*-modules in at least two ways, however the two C*-valued inner products are not unitarily isomorphic, and so the resulting two norms give rise to non-isometrically isomorphic Banach C*-modules, ([18]). Luckily, the latter oddity cannot appear for countably generated Hilbert C*-modules. For a concrete example consider the C*-algebra $A = C([0,1])$ of all continuous functions on the unit interval, its ideal $I = C_0((0,1])$ of all continuous functions vanishing at zero and the Hilbert A-module $\mathcal{M} = A \oplus I$ with the

standard A-valued inner product inherited from $\mathcal{M} \subset A^2$. The operator $T : (a, i) \to (i, 0)$ is non-adjointable, and the Hilbert A-submodule $\mathcal{N} = \{(i, i) : i \in I\}$ is a topological direct summand, but not an orthogonal one.

In the light of these circumstances the results on the existence and on the properties of modular frames for finitely or countably generated Hilbert C*-modules presented below become the more remarkable. We start with a definition of modular frames which takes an inequality in the positive cone of the C*-algebra of coefficients as its initial point.

DEFINITION 2.2. *Let A be a unital C*-algebra and $\mathbb{J}$ be a finite or countable index set. A sequence $\{x_j : j \in \mathbb{J}\}$ of elements in a Hilbert A-module $\mathcal{M}$ is said to be a frame if there are real constants $C, D > 0$ such that*

$$C \cdot \langle x, x \rangle \leq \sum_j \langle x, x_j \rangle \langle x_j, x \rangle \leq D \cdot \langle x, x \rangle \tag{1}$$

for every $x \in \mathcal{M}$. The optimal constants (i.e. maximal for C and minimal for D) are called frame bounds. The frame $\{x_j : j \in \mathbb{J}\}$ is said to be a tight frame if $C = D$, and said to be normalized if $C = D = 1$. We consider standard (normalized tight) frames in the main for which the sum in the middle of the inequality (1) always converges in norm. For non-standard frames the sum in the middle converges only weakly for at least one element of $\mathcal{M}$.

A sequence $\{x_j\}_j$ is said to be a standard Riesz basis of $\mathcal{M}$ if it is a standard frame and a generating set with the additional property that A-linear combinations $\sum_{j \in S} a_j x_j$ with coefficients $\{a_j\} \in A$ and $S \in \mathbb{J}$ are equal to zero if and only if in particular every summand $a_j x_j$ equals zero for $j \in S$. A generating sequence $\{x_j\}_j$ with the described additional property alone is called a Hilbert basis of $\mathcal{M}$.

An inner summand of a standard Riesz basis of a Hilbert A-module $\mathcal{L}$ is a sequence $\{x_j\}_j$ in a Hilbert A-module $\mathcal{M}$ for which there is a second sequence $\{y_j\}_j$ in a Hilbert A-module $\mathcal{N}$ such that $\mathcal{L} \cong \mathcal{M} \oplus \mathcal{N}$ and the sequence consisting of the pairwise orthogonal sums $\{x_j \oplus y_j\}_j$ in the Hilbert A-module $\mathcal{M} \oplus \mathcal{N}$ is the initial standard Riesz basis of $\mathcal{L}$.

Two frames $\{x_j\}_j$, $\{y_j\}_j$ of Hilbert A-modules H_1, H_2, respectively, are unitarily equivalent (resp., similar) if there exists a unitary (resp., invertible adjointable) linear operator $T : H_1 \to H_2$ such that $T(x_j) = y_j$ for every $j \in \mathbb{J}$.

Analyzing this definition we do not know whether a frame is a generating set, or not. This will turn out to hold only during our investigations. We observe that for every (normalized tight) frame $\{x_i\}_i$ of a Hilbert space H the sequence $\{1_A \otimes x_i\}_i$ is a standard (normalized tight) module frame of the Hilbert A-module $\mathcal{M} = A \otimes H$ with the same frame bounds. So standard modular frames exist in abundance in the canonical Hilbert A-modules. At the same time wavelet theorists see that the C*-algebra A opens up an additional degree of freedom for constructions and investigations. For the existence of standard modular frames in arbitrary finitely or countably generated Hilbert A-modules we obtained the following simple fact:

THEOREM 2.3. ([1,2])
For every A-linear partial isometry V on A^n (or $l_2(A)$) the image sequence $\{V(e_j)\}_j$ of the standard orthonormal basis $\{e_j\}_j$ is a standard normalized tight frame of the image $V(A^n)$ (or $V(l_2(A))$). Consequently, every algebraically finitely generated or countably generated Hilbert A-module $\mathcal{M}$ possesses a standard normalized tight frame since they can be embedded into these standard Hilbert A-modules as orthogonal summands.

PROBLEM 2.4. *Does every Hilbert C*-module admit a modular frame ?*

The main property of frames for Hilbert spaces is the existence of the reconstruction formula that allows a simple standard decomposition of every element of the spaces with respect to the frame. We found that almost all the results for the Hilbert space situation described in ([20]) can be recovered. Sometimes the way of proving is exceptional long, for example to show that modular Riesz bases $\{x_i\}_i$ that are normalized tight frames have to be orthogonal bases for which the values $\{\langle x_i, x_i \rangle\}_i$ are all projections. Let us first formulate the reconstruction formula for normalized tight frames without the restriction to be standard:

THEOREM 2.5. (Th. 4.1[1])
Let A be a unital C-algebra, $\mathcal{M}$ be a finitely or countably generated Hilbert A-module and $\{x_j\}_j$ be a normalized tight frame of $\mathcal{M}$. Then the reconstruction formula*

$$x = \sum_j \langle x, x_j \rangle x_j \tag{2}$$

3

holds for every $x \in \mathcal{M}$ in the sense of convergence w.r.t. the topology that is induced by the set of semi-norms $\{|f(\langle .,.\rangle)|^{1/2} : f \in A^*\}$. The sum converges always in norm if and only if the frame $\{x_j\}_j$ is standard.
Conversely, a finite set or a sequence $\{x_j\}_j$ satisfying the formula (2) for every $x \in \mathcal{M}$ is a normalized tight frame of $\mathcal{M}$.

For a proof we have to refer to ([1,2]) since the proof is to long to be reproduced here, and the statement for non-standard normalized tight frames is some kind of summary of the entire work done. With the experience on the possible oddities of Hilbert C*-module theory in comparison to Hilbert space theory the following crucial fact is surprising because of the generality in which it holds. The existence and the very good properties of the frame transform of standard frames give the chance to get far reaching results analogous to those in the Hilbert space situation. Again, the proof is more complicated than the known one in the classical Hilbert space case, cf. ([20]).

THEOREM 2.6. (Th. 4.2[1])
Let A be a unital C*-algebra, $\mathcal{M}$ be a finitely or countably generated Hilbert A-module and $\{x_j\}_j$ be a standard frame of $\mathcal{M}$. The frame transform of the frame $\{x_j\}_j$ is defined to be the map

$$\theta : \mathcal{M} \to l_2(A) \quad , \qquad \theta(x) = \{\langle x, x_j \rangle\}_j$$

that is bounded, A-linear, adjointable and fulfills $\theta^*(e_j) = x_j$ for a standard orthonormal basis $\{e_j\}_j$ of the Hilbert A-module $l_2(A)$ and all $j \in \mathbb{J}$.
Moreover, the image $\theta(\mathcal{M})$ is an orthogonal summand of $l_2(A)$. For normalized tight frames we additionally get $P(e_j) = \theta(x_j)$ for any $j \in \mathbb{J}$, and θ is an isometry in that case.

The frame transform θ is the proper tool for the description of standard frames.

THEOREM 2.7. (Th. 6.1[1])
Let A be a unital C*-algebra, $\mathcal{M}$ be a finitely or countably generated Hilbert A-module and $\{x_j\}_j$ be a standard frame of $\mathcal{M}$. Then the reconstruction formula
$$x = \sum_j \langle x, S(x_j) \rangle x_j$$

holds for every $x \in \mathcal{M}$ in the sense of norm convergence, where the operator $S := (\theta^*\theta)^{-1}$ is positive and invertible.

The sequence $\{S(x_j)\}_j$ is a standard frame again, the *canonical dual frame of* $\{x_j\}_j$. The operator S is called the *frame operator of* $\{x_j\}_j$ on $\mathcal{M}$. The two theorems on reconstruction above bring to light some key properties of frame sequences:

COROLLARY 2.8. (Th. 5.4, 5.5[1])
Every standard frame of a finitely or countably generated Hilbert A-module is a set of generators.
Every finite set of algebraic generators of a finitely generated Hilbert A-module is a (standard) frame.

COROLLARY 2.9. Every standard frame can be realized as the image of a standard normalized tight frame by an adjointable invertible bounded module operator. In particular, every standard Riesz basis is the image of an orthogonal Hilbert basis $\{x_i\}_i$ with projection-valued values $\{\langle x_i, x_i \rangle\}_i$ by an adjointable invertible bounded module operator.

Let us show that there exists countably generated Hilbert C*-modules without standard Riesz bases: let $A = C([0,1])$ be the C*-algebra of all continuous functions on the unit interval $[0,1]$ and consider its ideal and Hilbert A-submodule $\mathcal{M} = C_0([0,1])$ of all continuous functions vanishing at zero. The module $\mathcal{M}$ is countably generated by the Stone-Weierstrass theorem. So it admits normalized tight frames. However, if it would have a standard Riesz basis, then it would contain an orthogonal Hilbert basis $\{x_i\}_i$ with non-trivial projection-valued values $\{\langle x_i, x_i \rangle\}_i$. At the other side, the only projection contained in the range of the A-valued inner product is the zero element. So it does not contain any standard Riesz basis.

We close our considerations on the frame transform arising from standard frames with a statement on the relation between unitary equivalence (or similarity) of frames and the characteristics of the image of the frame transform θ.

THEOREM 2.10. ([1])
Let A be a unital C*-algebra and $\{x_j\}_j$ and $\{y_j\}_j$ be standard (normalized tight) frames of Hilbert A-modules $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. The following conditions are equivalent:

(i) The frames $\{x_j\}_j$ and $\{y_j\}_j$ are unitarily equivalent or similar.

4

(ii) *Their frame transforms θ_1 and θ_2 have the same range in $l_2(A)$.*

(iii) *The sums $\sum_j a_j x_j$ and $\sum_j a_j y_j$ equal zero for exactly the same Banach A-submodule of sequences $\{a_j\}_j$ of $l_2(A)$.*

These results look pretty much the same as for the Hilbert space situation what makes them easy to apply. However, the proofs are more difficult, and they required an extensive search for possible counterexamples and obstacles beforehand. There are also statements that do not transfer to Hilbert C*-modules. For example, standard Riesz bases of Hilbert C*-modules may have more than one dual frame because of the existence of zero-divisors in the C*-algebra of coefficients, see Example 6.4[1] and Corollary 6.6[19,20].

More results on disjointness and inner sums of frames, as well as on various kinds of frame decompositions can be found in ([2]). Our basic publications[1,2] also contain a number of illustrating examples and counterexamples we would like to refer to.

3. INVARIANTS OF FINITELY GENERATED PROJECTIVE C*-MODULES

The best way for the description of the inner structure of Hilbert spaces is the selection of a ((ortho-) normal) basis and the characterization of elements by their coordinates. The notion of a basis makes essentially use of the notion of linear independence of vectors. Turning to finitely generated projective C*-modules over a certain fixed unital C*-algebra A we are most often faced with the absence of a reasonable notion of 'A-linear independence' of sets of module elements, e.g. finite sets of algebraic generators. Also, very often we have a lot of non-isomorphic A-modules possessing generating sets with the same number of algebraic generators. The task is to find additional invariants for the distinction of projective A-modules in such situations.

Fortunately, any set of algebraic generators of a finitely generated projective C*-module $\mathcal{M}$ over a unital C*-algebra A is a frame, a fact shown by the authors[1] in 1998. Furthermore, for every set of algebraic generators $\{x_1, ..., x_k\}$ of $\mathcal{M}$ there exists an A-valued inner product $\langle .,. \rangle$ on $\mathcal{M}$ turning this set $\{x_1, ..., x_k\}$ into a normalized tight frame. We show that the knowledge of the values $\{\langle x_i, x_j \rangle : 1 \leq i \leq j \leq k\}$ turns out to be sufficient to describe the A-module $\mathcal{M}$ up to uniqueness. Note that the elements $\{x_1, ..., x_k\}$ need not to be (A-)linearly independent, in general.

THEOREM 3.1. *Let A be a unital C*-algebra and let $\{\mathcal{M}, \langle .,. \rangle_{\mathcal{M}}\}$ and $\{\mathcal{N}, \langle .,. \rangle_{\mathcal{N}}\}$ be two finitely generated Hilbert A-modules. Then the following conditions are equivalent:*

(i) *$\{\mathcal{M}, \|.\|_{\mathcal{M}}\}$ and $\{\mathcal{N}, \|.\|_{\mathcal{N}}\}$ are isometrically isomorphic as Banach A-modules.*

(ii) *$\{\mathcal{M}, \langle .,. \rangle_{\mathcal{M}}$ and $\{\mathcal{N}, \langle .,. \rangle_{\mathcal{N}}\}$ are unitarily isomorphic as Hilbert A-modules.*

(iii) *There are finite normalized tight frames $\{x_1, ..., x_k\}$ and $\{y_1, ..., y_l\}$ of $\mathcal{M}$ and $\mathcal{N}$, respectively, such that $k = l$, $x_i \neq 0$ and $y_i \neq 0$ for any $i = 1, ..., k$, and $\langle x_i, x_j \rangle_{\mathcal{M}} = \langle y_i, y_j \rangle_{\mathcal{N}}$ for any $1 \leq i \leq j \leq k$.*

Proof. The equivalence of the conditions (i) and (ii) has been shown for countably generated Hilbert A-modules in Theorem 4.1.[18] The implication (ii)→(iii) can be seen to hold setting $y_i = U(x_i)$ for the existing unitary operator $U : \mathcal{M} \to \mathcal{N}$ and for $i = 1, ..., k$. The demonstration of the inverse implication requires slightly more work. For the given normalized tight frames $\{x_1, ..., x_k\}$ and $\{y_1, ..., y_l\}$ of $\mathcal{M}$ and $\mathcal{N}$, respectively, we define a A-linear operator V by the rule $V(x_i) = y_i$, $i = 1, ..., k$. For this operator V we obtain the equalities

$$
\begin{aligned}
\langle V(x), y_j \rangle_{\mathcal{N}} &= \sum_{i=1}^{k} \langle V(x), y_i \rangle_{\mathcal{N}} \langle y_i, y_j \rangle_{\mathcal{N}} = \sum_{i=1}^{k} \left\langle \sum_{m=1}^{k} \langle x, x_m \rangle_{\mathcal{M}} V(x_m), y_i \right\rangle_{\mathcal{N}} \langle x_i, x_j \rangle_{\mathcal{M}} \\
&= \sum_{i=1}^{k} \sum_{m=1}^{k} \langle x, x_m \rangle_{\mathcal{M}} \langle x_m, x_i \rangle_{\mathcal{M}} \langle x_i, x_j \rangle_{\mathcal{M}} = \sum_{m=1}^{k} \langle x, x_m \rangle_{\mathcal{M}} \left\langle x_m, \sum_{i=1}^{k} \langle x_j, x_i \rangle_{\mathcal{M}} x_i \right\rangle_{\mathcal{M}} \\
&= \left\langle x, \sum_{m=1}^{k} \langle x_j, x_m \rangle_{\mathcal{M}} x_m \right\rangle_{\mathcal{M}} = \langle x, x_j \rangle_{\mathcal{M}}
\end{aligned}
$$

which hold for every $x \in \mathcal{M}$. Consequently,

$$\langle x, x \rangle_{\mathcal{M}} = \sum_{i=1}^{k} \langle x, x_i \rangle_{\mathcal{M}} \langle x_i, x \rangle_{\mathcal{M}} = \sum_{i=1}^{k} \langle V(x), y_i \rangle_{\mathcal{N}} \langle y_i, V(x) \rangle_{\mathcal{N}} = \langle V(x), V(x) \rangle_{\mathcal{N}}$$

for any $x \in \mathcal{M}$, and the operator V is unitary. $\square$

COROLLARY 3.2. *Every finitely generated projective A-module $\mathcal{M}$ over a unital C*-algebra A can be reconstructed up to isomorphism from the following data:*

(i) *A finite set of algebraic non-zero modular generators $\{x_1, ..., x_k\}$ of $\mathcal{M}$.*

(ii) *A symmetric $k \times k$ matrix (a_{ij}) of elements from A, where a_{ij} is supposed to be equal to $\langle x_i, x_j \rangle_0$ for $1 \leq i \leq j \leq k$ and for the (existing and unique) A-valued inner product $\langle ., . \rangle_0$ on $\mathcal{M}$ that turns the set of algebraic modular generators $\{x_1, ..., x_k\}$ into a normalized tight frame of the Hilbert A-module $\{\mathcal{M}, \langle ., . \rangle_0\}$.*

The number of elements in sets of algebraic modular generators of $\mathcal{M}$ has a minimum, and it suffices to consider sets of generators of minimal length. Then the modular invariants can be easier compared permuting the elements of the generating sets if necessary.

Proof. We have already pointed out that the set of algebraic generators $\{x_1, ..., x_k\}$ of $\mathcal{M}$ is a frame with respect to any A-valued inner product on $\mathcal{M}$ which turns $\mathcal{M}$ into a Hilbert A-module. That means the inequality

$$C \cdot \langle x, x \rangle \leq \sum_{i=1}^{k} \langle x, x_i \rangle \langle x_i, x \rangle \leq D \cdot \langle x, x \rangle$$

is satisfied for two finite positive real constants C, D and any $x \in \mathcal{M}$, see Theorem 5.9[1]. What is more, for any frame of $\mathcal{M}$ there exists another A-valued inner product $\langle ., . \rangle_0$ on $\mathcal{M}$ with respect to which it becomes normalized tight. The latter inner product is unique as shown by Corollary 4.3, Theorem 6.1[1] and Theorem 4.4[2]. So assertion (iv) of the previous theorem demonstrates the complete assertion. $\square$

We can say more in case the finitely generated Hilbert A-module contains a modular Riesz basis, i.e. a finite set of modular generators $\{x_1, ..., x_k\}$ such that the equality $0 = a_1 x_1 + ... + a_k x_k$ holds for certain coefficients $\{a_1, ..., a_k\} \subset A$ if and only if $a_i x_i = 0$ for any $i = 1, ..., k$. Obviously, a modular Riesz basis is minimal as a set of modular generators, i.e. we cannot drop any of its elements preserving the generating property. However, there can exist totally different Riesz bases for the same module that consist of less elements, cf. Example 1.1[1]. Note that the coefficients $\{a_1, ..., a_k\}$ can be non-trivial even if $a_i x_i = 0$ for any index i since every non-trivial C*-algebra A contains zero-divisors. Not every Hilbert C*-module with a normalized tight modular frame does possess a modular Riesz basis. For an example we refer to Example 2.4[2].

In case of finitely generated projective W*-modules (and therefore, in the case of Hilbertian modules over finite W*-algebras) we are in the pleasant situation that they always contain a modular Riesz basis by W. L. Paschke's Theorem 3.1[21]. Moreover, by spectral decomposition every element x of a Hilbert W*-module $\mathcal{M}$ has a carrier projection of $\langle x, x \rangle$ contained in the W*-algebra of coefficients A. So we can ascertain the following fact:

PROPOSITION 3.3. *Let $\mathcal{M}$ be a finitely generated projective A-module over a W*-algebra A that possesses two finite modular Riesz bases $\{x_1, ..., x_k\}$ and $\{y_1, ..., y_l\}$. Then there exists an $l \times k$ matrix $\mathbf{F} = (f_{ij})$, $i = 1, ..., l$, $, j = 1, ..., k$, with entries from A such that $y_i = \sum_{j=1}^{k} f_{ij} x_j$ for any $i = 1, ..., l$, and analogously, there exists a $k \times l$ matrix $\mathbf{G} = (g_{ji})$ with entries from A such that $x_j = \sum_{i=1}^{l} g_{ji} y_i$ for any $j = 1, ..., k$.*

Suppose the left carrier projections of f_{ij} and g_{ji} equal the carrier projections of $\langle y_i, y_i \rangle$ and $\langle x_j, x_j \rangle$, respectively, and the right carrier projection of f_{ij} and g_{ji} equal the carrier projections of $\langle x_j, x_j \rangle$ and $\langle y_i, y_i \rangle$, respectively. Then the matrices $\mathbf{F}$ and $\mathbf{G}$ are Moore-Penrose invertible in $M_{kl}(A)$ and $M_{lk}(A)$, respectively. The matrix $\mathbf{F}$ is the Moore-Penrose inverse of $\mathbf{G}$, and vice versa.

Proof. Since both the modular Riesz bases are sets of modular generators of $\mathcal{M}$ we obtain two A-valued (rectangular, w.l.o.g.) matrices $\mathbf{F} = (f_{ij})$ and $\mathbf{G} = (g_{ji})$ with $i = 1, ..., l$ and $j = 1, ..., k$ such that

$$y_i = \sum_{m=1}^{k} f_{im} x_m \quad , \quad x_j = \sum_{n=1}^{l} g_{jn} y_n \, .$$

6

Combining these two sets of equalities in both the possible ways we obtain

$$y_i = \sum_{n=1}^{l} \left(\sum_{m=1}^{k} f_{im} g_{mn} \right) y_n \quad , \quad x_j = \sum_{m=1}^{k} \left(\sum_{n=1}^{l} g_{jn} f_{nm} \right) x_m$$

for $i = 1, ..., l$, $j = 1, ..., k$. Now, since we deal with sets of coefficients $\{f_{ij}\}$ and $\{g_{ji}\}$ that are supposed to admit special carrier projections, the coefficients in front of the elements $\{y_n\}$ and $\{x_m\}$ at the right side can only take very specific values:

$$\sum_{m=1}^{k} f_{im} g_{mn} = \delta_{in} \cdot q_n \quad , \quad \sum_{n=1}^{l} g_{jn} f_{nm} = \delta_{jm} \cdot p_m \,,$$

where δ_{ij} is the Kronecker symbol, $p_m \in A$ is the carrier projection of $\langle x_m, x_m \rangle$ and $q_n \in A$ is the carrier projection of $\langle y_n, y_n \rangle$. So $\mathbf{F} \cdot \mathbf{G}$ and $\mathbf{G} \cdot \mathbf{F}$ are positive idempotent diagonal matrices with entries from A. The Moore-Penrose relations $\mathbf{F} \cdot \mathbf{G} \cdot \mathbf{F} = \mathbf{F}$, $\mathbf{G} \cdot \mathbf{F} \cdot \mathbf{G} = \mathbf{G}$, $(\mathbf{F} \cdot \mathbf{G})^* = \mathbf{F} \cdot \mathbf{G}$ and $(\mathbf{G} \cdot \mathbf{F})^* = \mathbf{G} \cdot \mathbf{F}$ turn out to be fulfilled. $\square$

In total we found a convinient way to characterize finitely generated C*-modules over unital C*-algebras up to modular isomorphism by a small amount of additional elements of the C*-algebra of coefficients derived from the set of algebraic generators and from the module structure.

4. AN OPERATOR-THEORETIC PROBLEM RESOLVED USING FRAME THEORY

One of the classical problems of operator theory is the following: given a (finite or infinite) sequence $\{b_i\}_{i=1}^{\infty}$ of bounded operators on a certain separable Hilbert space H fulfilling the equality $\mathrm{id}_H = \sum_{i=1}^{\infty} b_i^* b_i$, determine the nature of the operators $\{b_i\}_{i=1}^{\infty}$. A first account was found by R. V. Kadison and J. R. Ringrose in II.11.2.24[22] using dilation methods, i.e. enlarging the Hilbert space H and subsequently extending the operators. We will use modular frame methods to resolve this problem without changing the Hilbert space H. That way we demonstrate the power of the developed methods. Some particular examples complete the picture.

PROPOSITION 4.1. *Let $\{b_i\}_{i=1}^{\infty} \in B(l_2)$ be a sequence of elements with the property that $\mathrm{id}_{l_2} = \sum_{i=1}^{\infty} b_i^* b_i$ in the sense of weak convergence. (In particular, the sequence could be norm-convergent, or only finitely many elements could be unequal to zero.) Then there exists a projection $p \in B(l_2)$, $p \sim \mathrm{id}_{l_2}$ via $uu^* = p$, $u^* u = \mathrm{id}_{l_2}$, and a sequence of partial isometries $\{v_i\}_{i=1}^{\infty} \in B(l_2)$ such that:*

(i) $v_i v_i^* = \mathrm{id}_{l_2}$, $v_i^* v_i$ are projections in $B(l_2)$ similar to id_{l_2}, $(v_i^* v_i) \perp (v_k^* v_k)$ for any $i \neq k$, $\sum_{i=1}^{\infty} v_i^* v_i = \mathrm{id}_{l_2}$;

(ii) the equality $b_i = u_i(v_i u)$ is valid for every $i \in \mathbb{N}$ and partial isometries $\{u_i\}_{i=1}^{\infty} \subset B(l_2)$, each u_i connecting the left carrier projections of $v_i u$ and of b_i, respectively.

Proof. Let $A = B(l_2)$ be the set of all linear bounded operators on the Hilbert space l_2, and recall that $l_2 = \oplus_{i=1}^{\infty} (l_2)_{(i)}$ as a direct sum of copies of the Hilbert space l_2 itself. Consequently, there are projections $\{p_i\}_{i=1}^{\infty}$ such that $p_i \perp p_j$ for any $i \neq j$, $\sum_{i=1}^{\infty} p_i = \mathrm{id}_{l_2}$, and $p_i \sim \mathrm{id}_{l_2}$ via partial isometries $\{v_i\}_{i=1}^{\infty}$ with $v_i v_i^* = \mathrm{id}_{l_2}$ and $v_i^* v_i = p_i$.

Define $l_2(A)' := \{\{a_i\}_{i=1}^{\infty} : \sup_N \left\| \sum_{i=1}^{N} a_i a_i^* \right\| < \infty\}$, i.e. the set of all sequences of elements of A for which the series converges weakly in $A = B(l_2)$. It can be identified with the set of all bounded A-linear maps r of the Hilbert A-module $l_2(A)$ into A setting $a_i = r(e_i)$, where $e_i = (0, ..., 0, 1_{A,(i)}, 0, ...)$. Since $A = B(l_2)$ is a W*-algebra, the set $l_2(A)'$ becomes a self-dual Hilbert A-module ([21]).

In fact, $l_2(A)'$ is isometrically isomorphic to $A = B(l_2)$ as a Hilbert A-module. To see this fix an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ of $l_2(A)'$ and an orthonormal basis $\{v_i\}_{i=1}^{\infty}$ of A and define

$$l_2(A)' \to A \quad , \qquad \{a_i\}_{i=1}^{\infty} \to \sum_{i=1}^{\infty} a_i v_i \,;$$

$$A \to l_2(A)' \quad , \qquad a \to \{a v_i^*\}_{i=1}^{\infty} \,.$$

The freedom of choice for this isomorphism is the careful selection of the orthonormal bases of the self-dual Hilbert A-modules $l_2(A)'$ and A.

If the sequence $\{b_i\}_{i=1}^\infty \in B(l_2)$ is given as described above then it is a (possibly non-standard) normalized tight frame of the self-dual Hilbert A-module $A = B(l_2)$ since we have

$$a = a \cdot \mathrm{id}_{l_2} = a \sum_{i=1}^\infty b_i^* b_i = \sum_{i=1}^\infty \langle a, b_i \rangle_{B(l_2)} b_i$$

for every $a \in A = B(l_2)$. There exists a frame transform $\theta : B(l_2) \to l_2(B(l_2))'$ defined by $a \to \{\langle a, b_i \rangle_{B(l_2)}\}_{i=1}^\infty$. The image of θ is a direct summand and self-dual Hilbert A-submodule of $l_2(A)'$. Moreover, θ is an isometry. Continuing the isometry θ to an isometry $\theta' : B(l_2) \to B(l_2)$ using the isometric isomorphism between $l_2(A)'$ and A we obtain

$$\theta' : B(l_2) \to B(l_2) \quad, \qquad a \to \sum_{i=1}^\infty \langle a, b_i \rangle_{B(l_2)} v_i = \sum_{i=1}^\infty a b_i^* v_i$$

for any $a \in A$. The structure of direct orthogonal summands of the Hilbert $B(l_2)$-module $B(l_2)$ is well-known, they are all generated by multiplying $B(l_2)$ by a specific orthogonal projection from the right. So we can characterize the image of θ' in $B(l_2)$ as $B(l_2)p$ for some $p = p^2 \geq 0$ of $B(l_2)$. Note that $\theta'(\mathrm{id}_{l_2}) = p$ since the module generator id_{l_2} is mapped to the module generator p. Let $u \in B(l_2)$ be the isometry linking p to id_{l_2} with $uu^* = p$, $u^*u = \mathrm{id}_{l_2}$. Now, we establish some information on the adjoint operator of θ':

$$
\begin{aligned}
\langle a, b_i \rangle_{B(l_2)} &= \langle a, (\theta')^*(v_i) \rangle_{B(l_2)} = \langle \theta'(a), v_i \rangle_{B(l_2)} \\
&= \left(\sum_{j=1}^\infty \langle a, b_j \rangle_{B(l_2)} v_j \right) v_i^* = \left(\sum_{j=1}^\infty \langle a, b_j \rangle_{B(l_2)} v_j \right) p v_i^* \\
&= \left\langle a, \sum_{j=1}^\infty v_i p v_j^* b_j \right\rangle_{B(l_2)}.
\end{aligned}
$$

Consequently, we get the frame decomposition $b_i = \sum_{j=1}^\infty v_i p v_j^* b_j$ for any $i \in \mathbb{N}$. The frame coefficients $\{v_i p v_j^*\}_{j=1}^\infty$ may be not the optimal ones. By Prop. 6.6[1] we have the general inequality:

$$(v_i u)(v_i u)^* = v_i p v_i^* = \sum_{j=1}^\infty v_i p v_j^* v_j p v_i^* \geq \sum_{j=1}^\infty \langle b_i, b_j \rangle \langle b_j, b_i \rangle = b_i b_i^* \tag{3}$$

for any $i \in \mathbb{N}$. By Theorem 2.1[23] we obtain $b_i = u_i(v_i u)$ for some elements $\{u_i\}_{i=1}^\infty$ with $\|u_i\| \leq 1$ and

$$u_i = \mathrm{strong} - \lim b_i^*(\varepsilon + v_i p v_i^*)^{-1} v_i u = \mathrm{strong} - \lim b_i^* v_i u(\varepsilon + u^* p_i u)^{-1}.$$

Of course the root of $v_i p v_i^*$ seems to be selected in an artificial way, the element $v_i p$ would do the job as well. However, the following inequality gives some more information on the background of the choice made:

$$\mathrm{id}_{l_2} = \sum_{i=1}^\infty b_i^* b_i = \sum_{i=1}^\infty u^* v_i^* u_i^* u_i v_i u \leq \sum_{i=1}^\infty u^* v_i^* \|u_i\|^2 v_i u \leq u^* \left(\sum_{i=1}^\infty p v_i^* v_i p \right) u = \mathrm{id}_{l_2}.$$

Since the left end equals the right end and $b_i^* b_i \leq (v_i u)^*(uv_i)$ holds for every $i \in \mathbb{N}$ the equality $b_i^* b_i = (v_i u)^*(v_i u)$ turns out to be valid for every $i \in \mathbb{N}$. Consequently, the linking elements $\{u_i\}_{i=1}^\infty$ can be selected as partial isometries of $B(l_2)$ mapping the left carrier projection of $v_i u$ to the left carrier projection of b_i for each $i \in \mathbb{N}$ because $B(l_2)$ is a von Neumann algebra and any root of a given positive operator can be described this way. By the inequality (3) the left carrier projection of b_i has to be lower-equal than the left carrier projection of $v_i u$ for any $i \in \mathbb{N}$. $\square$

EXAMPLE 4.2. Suppose, the set of operators $\{b_i\}_{i=1}^\infty$ is a set of pairwise orthogonal (positive) projections $\{p_i\}_{i=1}^\infty$ defined on l_2. This is the simplest situation one can think of. Then $u = p = \mathrm{id}_{l_2}$ and $v_i = u_i = p_i$ for any index i. Furthermore, there is the classical situation of generators of Cuntz algebras $\mathcal{O}_n$ and $\mathcal{O}_\infty$, where the operators $\{b_i\}_{i=1}^\infty$ are partial isometries themselves, with the properties required above. Here $u = p = \mathrm{id}_{l_2}$ and $v_i = b_i$ for any index i. The partial isometries u_i equal to the left carrier projections of the partial isometries b_i in the given situation.

Generally speaking, the crucial rule is played by the projection p corresponding to the sequence $\{b_i\}_{i=1}^\infty$ via its frame transform, and by its partition $\{p_i p\}_{i=1}^\infty$ by a chain of pairwise orthogonal projections $\{p_i\}_{i=1}^\infty$ summing up to one and each being similar to the identity operator on l_2.

5. APPROXIMATION OF FRAMES BY (NORMALIZED) TIGHT ONES

In the present section we consider a question on the approximation of frames of Hilbert spaces H by (normalized) tight ones that is related to certain methods of orthogonalization and renormalization of Hilbert bases. We have to resort to Hilbert spaces instead of Hilbert C*-modules since the problem is too complex to be treated in full generality. We give comments on the more general setting whereever possible. Unfortunately, we did not find a final solution of the problem, rather we obtained hints to the complexity and difficulty of it. The partial results are nevertheless worth to be discussed. Generally speaking, most distance measures on sets of frame operators seem to have rather an L^∞-character than an L^2-character, exept the Hilbert-Schmidt norm. So the stressed for uniqueness of best approximating tight frames often cannot be obtained.

The following fundamental problem has been pointed out by R. Balan and the first author[27] in July 1999 summarizing earlier investigations:

PROBLEM 5.1. Are there distance measures on the set of frames of Hilbert subspaces K of H with respect to which a multiple of the normalized tight frame $\{S^{1/2}(x_i)\}_i$ is the closest (normalized) tight frame to the given frame $\{x_i\}_i$ of the Hilbert subspace $K \subseteq H$, or at least one of the closest (normalized) tight frames?
If there are other closest (normalized) tight frames with respect to the selected distance measures, do they span the same Hilbert subspaces of H? If not, how are the positions of these subspaces with respect to $K \subseteq H$?

Let $\{e_i\}_i$ be an orthonormal basis of the Hilbert space $l_2(\mathbb{C})$. For the analysis operator $T : K \to l_2(\mathbb{C})$, $T(x_i) = \{\langle x_i, e_i \rangle\}_i$, of a subspace frame $\{x_i\}_i$ of K there is a polar decomposition $T = VS^{-1/2}$, where $S = (T^*T)^{-1}$ denotes the frame operator. We can easily check that $S^{1/2}(x_i) = V^*(e_i)$ for any $i \in \mathbb{N}$. Also the projection $P : l_2(\mathbb{C}) \to TT^*(l_2(\mathbb{C}))$ plays an important role.

Looking through the literature there are two approaches to this problem, one due to R. Balan[24] and the other due to V. I. Paulsen, T. R. Tiballi[25] and the first author.[26] R. Balan[24] starts with the definition of three distance measures for pairs of frames: The frame $\{x_i\}_i$ of the Hilbert space H is said to be *quadratically close* to the frame $\{y_i\}_i$ of H if there exists a non-negative number C such that the inequality

$$\left\| \sum_i c_i(x_i - y_i) \right\| \leq C \cdot \left\| \sum_i c_i y_i \right\|$$

is satisfied. The infimum of all such constants C is denoted by $c(y, x)$. In general, if $C \geq c(y, x)$ then $C(1 - C)^{-1} \geq c(x, y)$, however this distance measure is not reflexive. Two frames $\{x_i\}_i$ and $\{y_i\}_i$ of a Hilbert space H are said to be *near* if $d(x, y) = \log(\max(c(x, y), c(y, x)) + 1) < \infty$. They are near if and only if they are similar, (Th. 2.4[24]). The distance measure $d(x, y)$ is an equivalence relation and fulfils the triangle inequality.

THEOREM 5.2. (Th. 3.1[24])
For a given frame $\{x_i\}_i$ of H with frame bounds C, D the distance measures admit their infima at

$$\min c(y, x) = \min c(x, y) = \frac{\sqrt{D} - \sqrt{C}}{\sqrt{D} + \sqrt{C}}, \quad \min d(x, y) = \frac{1}{4}(\log(D) - \log(C)).$$

These values are achieved by the tight frames

$$\left\{ \frac{\sqrt{C} + \sqrt{D}}{2} S^{1/2}(x_i) \right\}_i, \left\{ \frac{2\sqrt{CD}}{\sqrt{C} + \sqrt{D}} S^{1/2}(x_i) \right\}_i, \{ \sqrt[4]{CD} S^{1/2}(x_i)\}_i, \tag{4}$$

listed in the same order as the measures above. The solution may not be unique, in general, however any tight frame $\{y_i\}_i$ of H that achieves the minimum of one of the three distance measures $c(y, x)$, $c(x, y)$ and $d(x, y)$ is unitarily equivalent to the corresponding solutions listed above.

The difference of the connecting unitary operator and the product of minimal distance times either $S^{1/2}$ or $S^{-1/2}$ fulfils a certain measure-specific operator norm equality which can be found at Th. 3.1[24]. We point out that the first constant at (4) is the arithmetic mean of $\sqrt{C}$ and $\sqrt{D}$, the second one is their harmonic mean and the third one is their geometrical mean.

The results by V. I. Paulsen, T. R. Tiballi[25] and the first author[26] are of slightly different character, however the operator $(P - |T^*|)$ has to be Hilbert-Schmidt for their validity.

THEOREM 5.3. (Th. 2.3[26])
The operator $(P - |T^*|)$ is Hilbert-Schmidt if and only if the sum $\sum_{j=1}^{\infty} \|\mu_j - x_j\|^2$ is finite for at least one normalized tight frame $\{\mu_i\}_i$ of a Hilbert subspace L of H that is similar to $\{x_i\}_i$. In this situation the estimate

$$\sum\nolimits_{j=1}^{\infty} \|\mu_j - x_j\|^2 \geq \sum\nolimits_{j=1}^{\infty} \|S^{1/2}(x_j) - x_j\|^2 = \|(P - |T^*|)\|_{c_2}^2$$

is valid for every normalized tight frame $\{\mu_i\}_i$ of any Hilbert subspace L of H that is similar to $\{x_i\}_i$, where $\|\cdot\|_{c_2}$ denotes the Hilbert-Schmidt norm. (The left sum can be infinite for some choices of subspaces L and normalized tight frames $\{\mu_i\}_i$ for them.)
Equality appears if and only if $\mu_i = S^{1/2}(x_i)$ for any $i \in \mathbb{N}$. Consequently, the symmetric approximation of a frame $\{x_i\}_i$ in a Hilbert space $K \subseteq H$ is the normalized tight frame $\{S^{1/2}(x_i)\}_i$ spanning the same Hilbert subspace $L \equiv K$ of H and being similar to $\{x_i\}_i$ via the invertible operator $S^{-1/2}$.

Applying the theorem to appropriate Riesz bases $\{x_i\}_i$ the normalized tight frame $\{S^{1/2}(x_i)\}_i$ turns out to be the symmetric orthogonalization of this basis as discovered by P.-O. Löwdin[28] in 1948. This is why the denotation 'symmetric approximation' has been selected for the normalized tight frame $\{S^{1/2}(x_i)\}_i$.

The resultsin ([26]) generalize to modular frames of countably generated Hilbert C*-modules over commutative C*-algebras, whereas for respective Hilbert C*-modules over non-commutative C*-algebras the proofs cannot reproduced at several crucial places where commutativity of the C*-algebra of coefficients is essential. So the formulation of propositions on the non-commutative case is an open problem at present.

To overcome the difficulties with the non-commutativity of the C*-algebra of coefficients we consider distance-measures based on the various frame operators. The properties of these operators do not depend on the choice of the set of coefficients ([1,2]). One idea could be to consider the distance with respect to the operator norm of the difference of the orthogonal projections P_i onto the ranges of the frame transforms of two given frames $\{x_i\}_i$ and $\{y_i\}_i$. Unfortunately, two frames of a certain Hilbert space H are similar if and only if these projections coincide (Th. 7.2[1]). So we would only characterize classes of similar frames. The better idea is to consider the operator norm distance of the respective frame transforms T_x, T_y or of the respective frame operators S_x, S_y. The latter act as positive diagonalizable operators on the given Hilbert C*-module, whereas the former map it to the standard countably generated Hilbert C*-module $l_2(A)$.

For any tight frame $\{y_i\}_i$ of a Hilbert space H the corresponding frame operator S_y equals to the identity operator times the frame bound value. For a given frame $\{x_i\}_i$ of H with frame bounds C, D the closest positive multiple of the identity operator to the frame operator S_x is $(1/C + 1/D)/2 \cdot \mathrm{id}$. Unfortunately, every tight frame $\{y_i\}_i$ of H with frame bound $(1/C + 1/D)/2$ fulfils this condition, and the relative position of tight frames in H is of greater importance than one can express that way. However, we got a hint for the kind of factor to be used. Now, let us consider the norm of differences of frame transforms.

PROPOSITION 5.4. *Let $\{x_i\}_i$ be a frame of a certain Hilbert space H with frame bounds C, D, and let S be its frame operator. The frame transform T of the tight frame $\{(\sqrt{C} + \sqrt{D})/2 \cdot S^{1/2}(x_i)\}$ is the closest one w.r.t. the operator norm to the frame transform T_x of the given frame $\{x_i\}_i$ among all the frame transforms of positive multiples of the normalized tight frame $\{S^{1/2}(x_i)\}_i$.*

Proof. Denote the frame transform of the tight frame $\{\lambda \cdot S^{1/2}(x_i)\}_i$ by T_λ for $\lambda > 0$. Because the frame transforms T_λ and T all possess the same coisometries in their respective polar decompositions we obtain the equality $\|T_\lambda - T\| = \|\lambda \cdot \mathrm{id} - S^{-1/2}\|$. Since the inequality $\sqrt{C} \leq S^{-1/2} \leq \sqrt{D}$ is valid and both $\lambda \cdot \mathrm{id}$ and $S^{-1/2}$ are diagonalizable with a common set of eigenvectors that forms a basis of H we obtain

$$\|\lambda \cdot \mathrm{id} - S^{-1/2}\| = \max\left\{|\lambda - \mu_j| : \mu_j \text{ any eigenvalue of } S^{-1/2}\right\}.$$

The right expression is minimal if and only if λ is the arithmetic mean of the lower and the upper spectral bound $\sqrt{C}$ and $\sqrt{D}$ of the positive invertible operator $S^{-1/2}$. □

Suppose the frame $\{x_i\}_i$ of a certain Hilbert space H is fixed and possesses the frame bounds C and D. Let us consider tight frames $\{y_i\}_i$ of H with frame bound $(\sqrt{C} + \sqrt{D})/2$ for which the norm of the difference of their frame transform T_y and of the frame transform T of the tight frame $\{(\sqrt{C} + \sqrt{D})/2 \cdot S_x^{1/2}(x_i)\}_i$ is small. The next example shows that there are usually a lot of quite different tight frames of H with frame bound $(\sqrt{C} + \sqrt{D})/2$

the frame transforms of which realize the same norm distance to the frame transform T_x of the initial frame as the distinguished tight frame $\{(\sqrt{C} + \sqrt{D})/2 \cdot S_x^{1/2}(x_i)\}_i$ of H. We want to point out that the example works already for finite-dimensional Hilbert spaces and for frames with finitely many elements.

EXAMPLE 5.5. Let $H = l_2$ be a separable Hilbert space and fix its standard orthonormal basis $\{e_i\}_i$. Set $x_1 = e_1$, $x_2 = 3 \cdot e_2$ and $x_i = 2 \cdot e_i$ for $i \geq 3$. The resulting set $\{x_i\}_i$ is a Riesz basis of H with frame bounds $C = 1$ and $D = 9$. Since $(\sqrt{C} + \sqrt{D})/2 = 2$ we can consider other tight frames $\{y_i\}_i$ of H with the same frame bound 4 that are defined as $y_i = 2 \cdot e_i$ for $i \neq 3$ and $y_3 = 2\mathrm{e}^{\mathrm{i}\phi} \cdot e_3$ for some $\phi \in (-2 \cdot \arcsin(1/4), 2 \cdot \arcsin(1/4))$. Obviously, $\|T_x - T_y\| = \|S^{-1/2} - U\|$ for $T_x = V S_x^{-1/2}$ and $U = V^* T_y$, where U maps y_i to $\langle y_i, y_i \rangle e_i$, $i \in \mathbb{N}$ and V is the identity map. Since both $S^{-1/2}$ and U are normal, diagonalizable and commuting, with a basis consisting of common eigenvectors $\{e_i\}_i$, we can estimate

$$\|T_x - T_y\| = \max\{|\lambda_j - \mu_j| : j \in \mathbb{N}, \ S^{-1/2}(e_j) = \lambda_j e_j, \ U(e_j) = \mu_j e_j\}$$

see E. A. Azoff and C. Davis,[29] or K. R. Davidson.[30,31] The eigenvalues can be counted, they are $\lambda_i = \langle x_i, x_i \rangle^{1/2}$ and $\mu_i = \langle y_i, y_i \rangle^{1/2}$ for $i \neq 3$ and $\mu_3 = \mathrm{e}^{-\mathrm{i}\phi} \langle y_3, y_3 \rangle^{1/2}$. Taking the concrete values from the definitions of both the frames we obtain that the maximum of the difference of the corresponding eigenvalues is determined by the first two terms as long as $|\phi| < 2 \cdot \arcsin(1/4)$. So all these tight frames $\{y_i\}_i$ parametrized by $\phi \in (-2 \cdot \arcsin(1/4), 2 \cdot \arcsin(1/4))$ realize the same norm $\|T_x - T_y\| = 1$ for the difference of the respective frame transforms.

Summarizing, the measure of nearness of a frame to some tight frame derived from the norm of the difference of their frame transforms in general gives an entire manifold of tight frames that are 'closest' to a given frame with respect to this measure. Moreover, if we allow the tight frame to span a probably smaller Hilbert space than the original frame $\{x_i\}_i$, then the condition $D < 9/4 \cdot C$ to the frame bounds C, D of $\{x_i\}_i$ turns out to be occasionally essential to guarantee that the closest tight frame spans exactly the same Hilbert space than the initial frame. In other words, the distance between the square root of the lower frame bound C and the arithmetic mean of the square roots of the lower and the upper frame bounds C and D, respectively, has to be smaller than the distance of the square root of C to zero.

The problem stated in the beginning of the present section remains unsolved despite of the encouraging partial results indicated above, even for the approximation of frames of Hilbert spaces by (normalized) tight ones. We will continue our work to find a solution for it.

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